1. Dve enaki matematični nihali sta nameščeni na ekvatorju in na polu sferičnega planeta. Nihajni čas nihala na ekvatorju je 2 % daljši kot na polu. Če nihalo na polu dvignemo na višino 130 km, postanete njegov nihajni čas enako tistemu na ekvatorju. Planet se okoli svoje osi zavrti v 10 zemeljskih urah. S kakšno največjo hitrostjo se lahko premikate po površini takšnega planeta brez uporabe motorjev?

2. Na nebu sta vidni dve dvojni zvezdi, katerih komponente je mogoče razločiti v vidni svetlobi. Znano je, da sta od štirih zvezd, ki tvorijo te sisteme, dve pritlikavki, dve orjakinji, dve sta rdeči in dve modri. Orjakinji imata enak izsev in tudi pritlikavk imata enak izsev. Ugotovi, katere opisane zvezde tvorijo posamezno dvozvezdje oziroma so del istega sistema. Kateri od teh dveh sistemov je starejši?

3. Oceni najmanjšo oddaljenost, s katere vijolični premik črt v spektru galaksij ne bo viden.

4. Pri opazovanju dvozvezdja, katerega ena komponenta je bela pritlikavka, je bilo ugotovljeno, da se spektralna črta H_α razširjena na polovici intenzitete za 0,46 Angstroma. Hkrati je v območju svetlobe V zaznati zmanjšanje sija s periodo 0,5 leta. Kakšne so meje za maso spremljevalne zvezde bele pritlikavke? Dvozvezdje vidimo v orbitalni ravnini sistema, orbite so krožne.

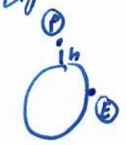
5. Leta 1961 so sovjetski inženirji pri izračunu orbit satelitov modelirali potencial sploščenosti Zemlje z uporabo dveh negravitacijskih mas (vsaka enaka polovici mase Zemlje), ki se nahajata na neki razdalji druga od druge vzdolž Zemljine vrtilne osi. Sodobni model potenciala Zemlje v tretjem približku je opisan s formulo:

$$V(r,\varphi) = \frac{GM_\oplus}{r}\left[1 - J_2\left(\frac{R_\oplus}{r}\right)^2 \frac{3\sin^2\varphi - 1}{2}\right]$$

kjer je r razdalja od središča Zemlje do dane točke, φ je zemljepisna širina, G je gravitacijska konstanta, $M_\oplus$ in $R_\oplus$ masa in polmer Zemlje, $J_2 \approx 1{,}08 \times 10^{-3}$ je koeficient. Izračunaj razdaljo med masama v modelu.

1.] ~~Comparing~~ Because the periods of two pendulums are the same, the gravitational acceleration is the same. 1/8

Peter Andolšek



On equator $g_e = g_{earth} + g_{centrifugal}$

On pole $g_p = g_{earth} + \underset{\uparrow}{\Delta} g$

due to the change of height h

Therefore,

$$g_e = g_p = g_{earth} - g_c = g_{earth} - \Delta g$$

$$g_c = \Delta g$$

$$\underline{\omega^2 r} = \frac{GM}{(r+h)^2} - \frac{GM}{r^2} = GM\left(\frac{1}{(r+h)^2} - \frac{1}{r^2}\right) = GM\left(\frac{\cancel{r^2} - \cancel{r^2} - 2rh - \overset{\text{negligible}}{h^2}}{r^2 \underset{\text{negligible}}{(r+h)^2}}\right) = \frac{2GMrh}{r^4} = \underline{\frac{2GMh}{r^3}}$$

$$g_0 = \frac{GM}{r^2} = \frac{\omega^2 r}{2} \cdot \frac{r}{h} = \frac{\omega^2 r^2}{2h}$$

$$\underset{\text{of pendulum}}{t_0} = 2\pi\sqrt{\frac{l}{g}}$$

$$\eta = \frac{t_e}{t_p} = 1{,}02 = \sqrt{\frac{g_p}{g_e}} = \sqrt{\frac{g_0}{g_0 - \omega^2 r}} = \sqrt{\frac{\omega^2 r^2}{2h\left(\frac{\omega^2 r^2}{2h} - \omega^2 r\right)}} =$$

$$= \sqrt{\frac{\cancel{\omega^2} r^{\cancel{2}}}{\cancel{\omega^2} r^{\cancel{2}} - 2h\cancel{\omega^2}\cancel{r}}} = \sqrt{\frac{r}{r-2h}} = \eta$$

$$\eta^2 r - 2\eta h = r$$

$$r = \frac{2\eta h}{\eta^2 - 1} = \frac{2 \cdot 1{,}02 \cdot 130\,\text{km}}{1{,}02^2 - 1} = \frac{260\,\text{km} \cdot 1{,}02}{4{,}04} \cdot 10^2 =$$

$$1{,}02^2 - 1 \to \frac{0{,}02 \cdot 2{,}02}{0{,}0404}$$

$$= \frac{260\,\text{km}}{4} \cdot 100 = \underline{\underline{6500\,\text{km}}}$$

And mass:

$$\omega^2 r^4 = 2GMh$$

$$M = \frac{\omega^2 r^4}{2Gh} =$$

$$= \frac{2^2 \cdot 6{,}5^3 \cdot 10^{24}}{10^2 \cdot 3600^2 \cdot 6{,}67 \cdot 10^{-11} \cdot 130 \cdot 10^3}\,\text{kg} = \frac{2 \cdot 6{,}5^3 \cdot 10^{24}}{3{,}6^2 \cdot 10^{-1} \cdot 13 \cdot 10}\,\text{kg} =$$

$$= \frac{2 \cdot 6{,}5^3}{3{,}6^2 \cdot 13} \cdot 10^{24}\,\text{kg} = \frac{16 \cdot 3{,}6}{13} \cdot 10^{24}\,\text{kg} = \underline{4 \cdot 10^{24}\,\text{kg}}$$

The data is very similar to Earth's.

Nevertheless, let's compare the ratios:

$$g_{earth} = \frac{6}{4} g_{planet}$$

The velocity of movement is proportional to the root of g.

Hence, $v_{planet} = 1{,}2 \cdot \underset{\approx 20\frac{\text{km}}{\text{h}}}{\underbrace{v_{earth}}} = \underline{\underline{24\,\frac{\text{km}}{\text{h}}}}$

2/8
Peter Andolšek

2. The dwarf stars cannot be red, therefore:

Both dwarfs: blue $\rightarrow \approx$ same temperature
Both giants: red $\rightarrow \approx$ same temperature

Because their luminosity is the same, and they have same temperature,

Both dwarfs are identical.
Both giants are identical.

~~Most~~ The most probable combination is dwarf — giant. (they are least rare)
Since, both stars in particular category are the same, the age of the system in this configuration is impossible to give.

~~...~~ On the other hand, since the problem asks us for the age, this might mean that the age is identifiable, and the configuration is:

dwarf – dwarf   giant – giant

In that case, the system dwarf–dwarf is older, since their ancestor stars had already died.

3.] The red/blueshift of the galaxy's spectrum consists of two sources: 3/8

Peter Andolšek

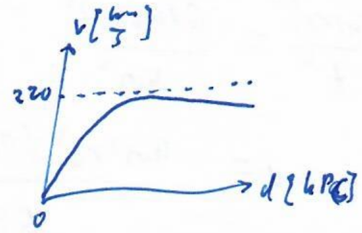
Rotation of galaxy + Expansion of the Universe.

For the redshift to be equal to zero, the maximum rotational speed needs to be equal to that of expansion of the Universe.

Using Hubble-Lemaître law,

$v_{exp} = H_0 d$

$d = \frac{v_{exp}}{H_0} = \frac{v_{rot}}{H_0}$

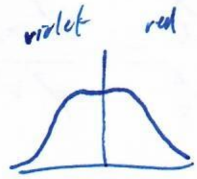


Typical maximum rotational velocity is $v_{rot} \approx 220 \frac{km}{s}$

Therefore,

$$d = \frac{v_{rot}}{H_0} = \frac{220 \frac{km}{s}}{70 \frac{km}{s}} Mpc = \underline{3,1\ Mpc}$$

Which is also the answer.



In the stationary galaxy, lines are moved to 2 sides:

But in moving galaxy, they are systematically moved to the red part:

Explanation.

4/8

Peter Andolich

4.

$$\frac{G(m_d+m_n)}{(r_n+r_d)^2} = \frac{v^2}{r_n} = \frac{G(m_d+m_n)}{r_n^2\left(1+\frac{m_n}{m_d}\right)^2} = \frac{v^2}{r_n}$$

$\frac{v}{c} = \frac{\Delta\lambda}{\lambda}$; $\lambda = 6600$ Å

$v = 21 \frac{km}{s}$

$\cancel{v = \frac{2\pi r_n}{t}}$ $\cancel{r_n = \frac{vt}{2\pi}}$

barycenter ($m_d r_d = m_n r_n$)

$$\frac{G(m_d+m_n)}{r_n\left(1+\frac{m_n}{m_d}\right)^2} = v^2 = \frac{G(m_d+m_n)\,m_d^2}{r_n(m_d+m_n)^2} = v^2 =$$

$$= \boxed{v^2 = \frac{G m_d^2}{r_n(m_d+m_n)}}$$

3rd law:

$$\frac{(r_n+r_d)^3}{t^2} = \frac{G(m_n+m_d)}{4\pi^2}$$

$$m_n+m_d = \frac{4\pi^2 r_n^3\left(1+\frac{m_n}{m_d}\right)^3}{G t_0^2} = \frac{4\pi^2 r_n^3 (m_n+m_d)^{\cancel{3}\,2}}{G t_0^2 m_d^3} = \cancel{m_n+m_d}$$

$$\boxed{m_n+m_d = \sqrt{\frac{G t_0^2 m_d^3}{4\pi^2 r_n^3}}}$$

$$m_d+m_n = \frac{G m_d^2}{r_n v^2}$$

$$\frac{\cancel{G} t_0^2 m_d^3}{4\pi^2 r_n^3} = \frac{G^2 m_d^4}{r_n^2 v^4}$$

$$\frac{t_0^2}{4\pi^2 r_n} = \frac{G m_d}{v^4} \Big\} \rightarrow r_n = \frac{t_0^2 v^4}{4\pi^2 G m_d}$$

$$m_n = \frac{G m_d^2}{r_n v^2} - m_d = \frac{G^2 m_d^3 \cdot 4\pi^2}{t_0^2 v^6} - m_d = \frac{4\pi^2 G^2 m_d^3}{t_0^2 v^6} - m_d = m_n$$

$m_d \in (0{,}2 \; ; \; 1{,}3)\, M_\odot$

See auxillary calculations for calculations.

$$\frac{4\pi^2 \cdot 6{,}67^2 \cdot 10^{-22} \cdot m^3 \cdot 2^3 \cdot 10^{90}}{2 \cdot 0{,}5^2 \cdot 86400^2 \cdot 365^2 \cdot 21^6 \cdot 10^{18}} - m \cdot 2 \cdot 10^{30} = 7{,}5 \cdot 10^{31} \cdot m^3 - 2 \cdot 10^{30} \cdot m$$

For $m_d = 0{,}2 M_\odot \rightarrow m_n = 2 \cdot 10^{29}$ kg

$m_d = 1{,}3 M_\odot \rightarrow m_n = 1{,}6 \cdot 10^{32}$ kg

$\Big\}\; 2 \cdot 10^{29}\,\text{kg} < m_n < 1{,}6 \cdot 10^{32}\,\text{kg}$

5.

Peter Andolšek

$$V(r,\varphi) = \mp \frac{GM}{r}\left[1 \pm J_2 \left(\frac{R}{r}\right)^2 \frac{3\sin^2\varphi - 1}{2}\right]$$

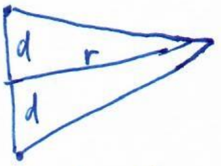
For an 1961 satellite, $R = r$ (they were not yet good enough to fly any further than $h = 400\,km$; $r = R + h \doteq R$)

Also, they circle around equator. Hence, $\varphi = 0°$

Our simplified potential:

$$V = \mp \frac{GM}{r}\left[1 \mp \frac{J_2}{2}\right]$$

Consider now two spheres:



Distance to them is clearly $\sqrt{d^2+r^2}$

Therefore,

$$V = \frac{GM}{2\sqrt{d^2+r^2}} + \frac{GM}{2\sqrt{d^2+r^2}} = \frac{GM}{\sqrt{d^2+r^2}}$$

where we have used the fact that the mass of the sphere is $\frac{M}{2}$

$$\frac{\cancel{GM}}{\sqrt{d^2+r^2}} = \frac{\cancel{GM}}{r}\left(1 - \frac{J_2}{2}\right) = \frac{\cancel{GM}}{r}\left(\frac{2-J_2}{2}\right)$$

$$d^2 = r^2\left(\frac{2}{2-J_2}\right)^2 - r^2$$

$$d = r\sqrt{\left(\frac{2}{2-J_2}\right)^2 - 1} = 6400\,km \sqrt{\left(\frac{2}{2-1{,}08\cdot 10^{-3}}\right)^2 - 1} = 6400\,km\sqrt{0{,}00107} = 6400\,km\sqrt{10{,}7}\cdot 10^{-2} = 64\,km \cdot 3{,}27 = 209{,}38\,km \approx 209\,km$$

$$\frac{4}{4 + \cancel{1{,}08^2 \cdot 10^{-6}} - 2\cdot 2\cdot 1{,}08\cdot 10^{-3}} = \frac{4}{4 - 4{,}32\cdot 10^{-3}} = \frac{4}{3{,}99568} =$$

$40000 : 39957 = 1{,}00107\ldots$

-39957

00430

43000

39957

304300

$\ldots$

$3{,}3\cdot 3{,}3 = 10{,}89$

$3{,}25 \cdot 3{,}25 = 10{,}5625$

$\to 3{,}27$

$64\,km \cdot 3{,}27$: 192, 128, 458, $209{,}38$

$d = 209\,km$

The distance between spheres is then,

$2d = \underline{\underline{418\,km}}$

Peter Andolsek

$t_p = 2\pi\sqrt{\frac{l}{g_{eff}}}$

~~Sky~~ Auxidliary calculations

$g_{eff} = g_0 - \omega^2 r$

$g_c = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r$

$$\frac{t_{pe}}{t_{pp}} = 1{,}02 = \sqrt{\frac{g_{Pe\,eff}}{g_{pe\,eff}}} = \sqrt{\frac{g_0}{g_0 - \omega^2 r}} = \sqrt{\frac{\frac{GM}{r}}{r\left(\frac{GM}{r} - \omega^2 r\right)}} =$$

GM

$$= \sqrt{\frac{GM}{GM - \omega^2 r^2}} = 1{,}02$$

$$\frac{GM}{r+\Delta r} - \omega^2 (r+\Delta r)^2 = \frac{GM}{r} = g_0$$

$$GMr - \omega^2 (r+\Delta r)^2 r = GM(r+\Delta r)$$

$$\eta = 1{,}02 = \sqrt{\frac{GM - \omega^2 (r+\Delta r)^2}{(r+\Delta r)\left(\frac{GM - \omega^2 (r+\Delta r)^2}{r+\Delta r} - \omega^2 r\right)}} =$$

$$= \sqrt{\frac{GM - \omega^2 (r+\Delta r)^2}{GM - \omega^2 (r+\Delta r)^2 - \omega^2 r (r+\Delta r)}}$$

$$\eta^2 \left(GM - \omega^2 (r+\Delta r)^2 - \omega^2 r (r+\Delta r)\right) = GM - \omega^2 (r+\Delta r)^2$$

$g' = \frac{GM}{(r+h)^2} = \frac{GM}{r^2} + x$

$x = GM\left(\frac{1}{(r+h)^2} - \frac{1}{r^2}\right) = \frac{-2GMrh}{r^2(r+h)^2} = \frac{2GMrh}{r^4} = \frac{2GMh}{r^3}$

$g_0 = \frac{GM}{r^2}$

$= \frac{\omega^2 r^2}{h}$

$\frac{GMh}{r^3} = \omega^2 r$

$\boxed{GMh = \omega^2 r^4}$

$\frac{\omega^2 r^2}{h\left(\frac{\omega^2 r^2}{h} - \omega^2 r\right)} = \eta^2$

$\frac{r^2}{\frac{hr^2}{h} - rh} = \eta^2$

$\frac{r^2}{r^2 - rh} = \eta^2$ $\quad \frac{r}{r-h} = \eta^2$

$r = \eta^2 r - \eta^2 h$

$r = \frac{\eta^2 h}{\eta^2 - 1}$

$M = \frac{\omega^2 r^4}{Gh}$

$M = 3{,}4 \cdot 10^{23}\,kg$

$r = 3350\,km$

$\omega = 1{,}75 \cdot 10^{-4}\,s^{-1}$

$g = 2{,}62\,\frac{m}{s^2}$

$g_c = 0{,}1\,\frac{m}{s^2}$

$g_{eff} = 2{,}52\,\frac{m}{s^2}$

7/8 Peter Andolsch

$$V_S = \frac{GM}{2(r-d)} + \frac{GM}{2(r+d)} = \frac{GM2}{r}\left[1 - J_2\left(\frac{R}{r}\right)^2 \frac{3\sin^2\varphi - 1}{2}\right]$$

$$\frac{1}{r-d} + \frac{1}{r+d} = \frac{2}{r}\left[1 + \frac{J_2}{2}\left(\frac{R}{r}\right)^2\right] = \frac{2}{r} + J_2\frac{R^2}{r^3}$$

$$\frac{r+d+r-d}{r^2-d^2} = \frac{2r}{r^2-d^2} = \frac{2}{r} + \frac{J_2R^2}{r^3} = \frac{2r^2+J_2R^2}{r^4} = \frac{2}{r^2-d^2}$$

$$r^2-d^2 = \frac{2r^4}{2r^2+J_2R^2} = \frac{2r^4}{2r^2+J_2r^2} = \frac{2r^2}{2+J_2}$$

$\rightarrow r = R$

$d = 150$ km

$$\frac{1}{2r_1} + \frac{1}{2r_2} =$$

$$= \frac{\sqrt{r^2+d^2-2rd\sin\varphi} + \sqrt{r^2+d^2+2rd\sin\varphi}}{2\sqrt{(r^2+d^2)^2 - 4r^2d^2\sin^2\varphi}} = \frac{1}{r}\left(1 - J_2\frac{3\sin^2\varphi - 1}{2}\right)$$

$r_1^2 = r^2 + d^2 - 2rd\cos(\varphi+90) = r^2+d^2+2rd\sin\varphi$

$r_2^2 = r^2+d^2-2rd\sin\varphi$

$r_1 = \sqrt{r^2+d^2+2rd\sin\varphi}$

$r_2 = \sqrt{r^2+d^2-2rd\sin\varphi}$

$$\frac{1}{r+d} + \frac{1}{r-d} = \frac{r-d+r+d}{r^2-d^2} = \frac{2r}{r^2-d^2}$$

$$\boxed{\frac{v}{c} = \frac{\Delta\lambda}{\lambda}}$$

$0{,}2 - 1{,}3$

$$\frac{G(m_1+m_2)}{4\pi^2} = \frac{a^3}{t_0^2}$$

$$t_0^2 = \frac{4\pi^2 r^3}{GM}$$

$v = 21\frac{km}{s}$

$$\frac{GM}{r^2} = \frac{v^2}{r}$$

$$\frac{GM}{r} = v^2$$

$$r = \frac{GM}{v^2}$$

$$\frac{a_3}{t_0^2} = \frac{G(m_p+m_d)}{4\pi^2}$$

$GM = v^2 r$

$$GM = \frac{4\pi^2 r^3}{t_0^2}$$

$$1 = \frac{v^2 r t_0^2}{4\pi^2 r^3}$$

$4\pi^2 r^3 = v^2 r t_0^2$

$4\pi^2 r^2 = v^2 t_0^2$

$$v = \frac{2\pi r}{t_0}$$

$$\boxed{r = \frac{v t_0}{2\pi}} = 5{,}3\cdot 10^{10}\,\mathrm{m} = 0{,}35\,\mathrm{AU}$$

$GM = v^2 r$

$M = 4\cdot 10^{?}$ kg

$m_d \cdot r_d = m_n \cdot r_n$

$$r_d = \frac{m_n r_n}{m_d}$$

$$\frac{G(m_d+m_n)}{r_n+r_d} = v^2$$

$$\frac{Gm_d+Gm_n}{r_n\left(1+\frac{m_n}{m_d}\right)} = v^2$$

$$\frac{G(m_d+m_n)m_d}{r_n(m_d+m_n)} = v^2$$

$$\frac{Gm_d}{r_n} = v^2$$

8/8
Peter Andolšek

$$\frac{G(m_d+m_n)}{(r_n+r_d)^2} = \frac{v^2}{r_n} = \frac{G(m_d+m_n)}{r_n^2\left(1+\frac{m_n}{m_d}\right)^2} = \frac{v^2}{r_n} = \frac{G(m_d+m_n)m_d^2}{r_n(m_d+m_n)^2} = \boxed{\frac{G}{r_n}\,\frac{m_d^2}{m_d+m_n} = v^2}$$

$$\frac{m^3}{kg\,s^2}\,\frac{kg}{m}$$

$$\cancel{m_d+}m_n = \frac{4\pi^2 (r_n+r_d)^3}{G t_0^2} = \frac{4\pi^2 r_n^3\left(1+\frac{m_n}{m_d}\right)^3}{G t_0^2} = \frac{4\pi^2 r_n^3 (m_n+m_d)^3}{G t_0^2 m_d^3}$$

$$m_d+m_n = \frac{G m_d^2}{r_n v^2} = \frac{G t_0^2 m_d^3}{4\pi^2 r_n^3}$$

$$G t_0^2 m_d^3 = 4\pi^2 r_n^3 (m_n+m_d)$$

$$\frac{1}{v^2} = \frac{t_0^2 m_d}{4\pi^2 r_n^2}$$

$$r_n^2 = \frac{t_0^2 m_d v^2}{4\pi^2}$$

$$r_n = \frac{t_0 v}{2\pi}\sqrt{m_d}$$

$$\frac{2\pi G}{t_0 v\sqrt{m_d}}\,\frac{m_d^{3/2}}{m_d+m_n} = v^3$$

$$\frac{2\pi G m_d^{3/2}}{t_0 v^3} - m_d = m_n$$